

ON THE EQUILIBRIUM CONFIGURATION OF THE BC-TYPE RUIJSENAARS-SCHNEIDER SYSTEM

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ABSTRACT. It is shown that the ground-state equilibrium configurations of the trigonometric BC -type Ruijsenaars-Schneider systems are given by the zeros of Askey-Wilson polynomials.

1. INTRODUCTION

The Ruijsenaars-Schneider systems [12, 5, 6] are integrable deformations of the celebrated Calogero-Moser n -particle models [10]. It is well-known that the equilibrium configurations of the Calogero-Moser models are described by the zeros of classical orthogonal polynomials such as the Hermite, Laguerre, Chebyshev, and Jacobi polynomials [2, 3, 4, 10]. This connection between the equilibria of one-dimensional integrable particle models and the locations of zeros of the classical hypergeometric orthogonal polynomials, first observed by Calogero, is closely related to a beautiful electrostatic interpretation of the zeros of orthogonal polynomials due to Stieltjes [14]. Recently, it was noticed that the equilibrium configurations of the Ruijsenaars-Schneider systems can also be described in a similar way by means of the zeros orthogonal polynomials [13, 11, 9]; all polynomials that appear in this context turn out to be classical in the sense that they sit somewhere in Askey's hierarchy of (basic) hypergeometric orthogonal polynomials [1, 8]. The top of this hierarchy is formed by the Askey-Wilson polynomials [1]. (All other (basic) hypergeometric families of classical orthogonal polynomials are special (limiting) cases of the Askey-Wilson polynomials [8].) In this note we show that the zeros of these Askey-Wilson polynomials correspond to the ground-state equilibrium configurations of the trigonometric BC -type Ruijsenaars-Schneider systems introduced in Refs. [5, 6]. In the rational limit, one recovers the characterization of the ground-state equilibrium configurations of the rational BC -type Ruijsenaars-Schneider systems in terms of Wilson polynomials [15] due to Otake and Sasaki [9].

We will need to employ the following standard conventions from the theory of (basic) hypergeometric orthogonal polynomials [1, 8]: q -shifted factorials are denoted by

$$(a; q)_k := \begin{cases} 1 & \text{for } k = 0, \\ (1-a)(1-aq) \cdots (1-aq^{k-1}) & \text{for } k = 1, 2, 3, \dots, \end{cases}$$

with the convention that $(a; q)_\infty := \prod_{k=0}^{\infty} (1-aq^k)$ (for $|q| < 1$); products of q -shifted factorials are abbreviated in the usual way via

$$(a_1, \dots, a_r; q)_k := (a_1; q)_k \cdots (a_r; q)_k.$$

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The ${}_{r+1}\Phi_r$ terminating basic hypergeometric series is defined as

$${}_{r+1}\Phi_r \left[\begin{matrix} q^{-n}, a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} \mid q; z \right] := \sum_{k=0}^n \frac{(q^{-n}, a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_r; q)_k} z^k$$

(where it is assumed that the parameters are such that denominators do not vanish). The hypergeometric degeneration of this series is given by

$${}_{r+1}F_r \left[\begin{matrix} -n, a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix} \mid z \right] := \sum_{k=0}^n \frac{(-n, a_1, \dots, a_r)_k}{(1, b_1, \dots, b_r; q)_k} z^k,$$

where $(a_1, \dots, a_r)_k := (a_1)_k \cdots (a_r)_k$ with $(a)_k := a(a+1) \cdots (a+k-1)$ (and $(a)_0 := 1$ by convention).

2. THE TRIGONOMETRIC BC -TYPE RUIJSENAARS-SCHNEIDER SYSTEM

The trigonometric BC -type Ruijsenaars-Schneider system is a one-dimensional n -particle model characterized by the Hamiltonian [5, 6]

$$H(\mathbf{p}, \mathbf{x}) = \sum_{j=1}^n \left(\cosh(p_j) \sqrt{V_j(\mathbf{x})V_j(-\mathbf{x})} - (V_j(\mathbf{x}) + V_j(-\mathbf{x}))/2 \right), \quad (2.1a)$$

where

$$V_j(\mathbf{x}) = w(x_j) \prod_{1 \leq k \leq n, k \neq j} v(x_j + x_k) v(x_j - x_k), \quad (2.1b)$$

$$v(x) = \frac{\sin(x + ig)}{\sin(x)}, \quad w(x) = \frac{\sin(x + ig_1) \cos(x + ig_2) \sin(x + ig_3) \cos(x + ig_4)}{\sin^2(x) \cos^2(x)} \quad (2.1c)$$

(and $i := \sqrt{-1}$). Throughout this note we will assume that the coupling parameters g and g_r ($r = 1, 2, 3, 4$) are *positive*. This guarantees in particular that the Hamiltonian $H(\mathbf{p}, \mathbf{x})$ (2.1a)-(2.1c) constitutes a nonnegative (smooth) function on the phase space

$$\Omega = \{(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^{2n} \mid 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n < \pi/2\}. \quad (2.2)$$

Indeed, one has that $H(\mathbf{p}, \mathbf{x}) \geq H(\mathbf{0}, \mathbf{x}) \geq 0$ (since $V_j(-\mathbf{x}) = \overline{V_j(\mathbf{x})}$ and $|V_j(\mathbf{x})| \geq \operatorname{Re}(V_j(\mathbf{x}))$).

3. THE GROUND-STATE EQUILIBRIUM CONFIGURATION

The equilibrium configurations correspond to the critical points of the Hamiltonian $H(\mathbf{p}, \mathbf{x})$ and the ground-state equilibrium configurations correspond in turn to the global minima. It is clear that the only way in which the nonnegative BC -type Ruijsenaars-Schneider Hamiltonian $H(\mathbf{p}, \mathbf{x})$ (2.1a)-(2.1c) may vanish (thus actually reaching the lower bound zero) is when $\mathbf{p} = \mathbf{0}$ and \mathbf{x} is such that $V_j(\mathbf{x})$ is positive for $j = 1, \dots, n$. This requires in particular that $V_j(\mathbf{x})$ is real-valued, i.e. that

$$V_j(\mathbf{x}) = V_j(-\mathbf{x}), \quad j = 1, \dots, n, \quad (3.1a)$$

or more explicitly

$$\prod_{1 \leq k \leq n, k \neq j} \frac{\sin(x_j + x_k + ig) \sin(x_j - x_k + ig)}{\sin(x_j + x_k - ig) \sin(x_j - x_k - ig)} = \quad (3.1b)$$

$$\frac{\sin(x_j - ig_1) \cos(x_j - ig_2) \sin(x_j - ig_3) \cos(x_j - ig_4)}{\sin(x_j + ig_1) \cos(x_j + ig_2) \sin(x_j + ig_3) \cos(x_j + ig_4)}, \quad j = 1, \dots, n.$$

We will see below that the nonlinear system of algebraic equations in Eq. (3.1b) has a unique solution $0 < x_1 < x_2 < \dots < x_n < \pi/2$ given by the zeros of the Askey-Wilson polynomial of degree n . It is not difficult to see that for this solution in fact $V_j(\mathbf{x}) > 0$ for $j = 1, \dots, n$, whence $H(\mathbf{0}, \mathbf{x}) = 0$. Indeed, $V_j(\mathbf{x})$ is real-valued by Eq. (3.1a). Furthermore, for sufficiently small values of the coupling parameters $V_j(\mathbf{x})$ must be positive as the function in question tends to 1 for $g, g_r \rightarrow 0$. This positivity remains valid for general positive parameter values g, g_r by a continuity argument revealing that the sign cannot flip (as none of the factors in $V_j(\mathbf{x})$ (2.1b), (2.1c) becomes zero or singular).

We thus arrive at the following theorem.

Theorem 3.1. *The trigonometric BC-type Ruijsenaars-Schneider Hamiltonian $H(\mathbf{p}, \mathbf{x})$ (2.1a)-(2.1c) assumes the global minimum $H = 0$ only at the point in the phase space Ω (2.2) such that $p_1 = p_2 = \dots = p_n = 0$ and $0 < x_1 < x_2 < \dots < x_n < \pi/2$ form a solution of the nonlinear system of algebraic equations in Eq. (3.1b).*

4. ZEROS OF THE ASKEY-WILSON POLYNOMIALS

The nonlinear system of algebraic equations in Eq. (3.1b) turns out to be a special case of the Bethe Ansatz equations associated to q -Sturm-Liouville problems studied recently by Ismail *et al* [7]. It follows from the machinery in *loc. cit.* that this algebraic system has a unique solution given by the zeros of the Askey-Wilson polynomial of degree n . Below we will provide an independent direct proof of this fact.

To this end, we first need to recall some basic properties of the Askey-Wilson polynomials taken from Ref. [1]. The (monic) Askey-Wilson polynomials are trigonometric polynomials of the form

$$p_n(x) = \cos(2nx) + \sum_{k=0}^{n-1} a_k \cos(2kx), \quad n = 0, 1, 2, \dots, \quad (4.1a)$$

obtained by applying Gram-Schmidt orthogonalization of the standard Fourier cosine basis $1, \cos(2x), \cos(4x), \dots$ on the interval $(0, \pi/2)$ with respect to the inner product

$$\langle f, g \rangle_\Delta = \int_0^{\pi/2} f(x) \overline{g(x)} \Delta(x) dx, \quad (4.1b)$$

associated to the weight function

$$\Delta(x) = \frac{1}{c(x)c(-x)}, \quad c(x) = \frac{(ae^{2ix}, b^{2ix}, ce^{2ix}, de^{2ix}; q)_\infty}{(e^{4ix}; q)_\infty}. \quad (4.1c)$$

Here it is assumed that all parameters are real-valued subject to the constraints $0 < q < 1$ and $0 < |a|, |b|, |c|, |d| < 1$. These parameter restrictions ensure in particular that the weight function $\Delta(x)$ (4.1c) is positive in the interval $(0, \pi/2)$.

The Askey-Wilson polynomials admit an explicit representation in terms of the following terminating basic hypergeometric series

$$p_n(x) = \frac{(ab, ac, ad; q)_n}{2a^n(abcdq^{n-1}; q)_n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{2ix}, ae^{-2ix} \\ ab, ac, ad \end{matrix} \middle| q; q \right]. \quad (4.2)$$

The polynomials under consideration are the eigenfunctions of a second-order difference operator. Upon performing the parameter substitution

$$q = e^{-2g}, \quad a = e^{-2g_1}, \quad b = -e^{-2g_2}, \quad c = e^{-2g_3}, \quad d = -e^{-2g_4}, \quad (4.3)$$

the corresponding eigenvalue equation becomes of the form

$$Dp_n(x) = E_np_n(x), \quad (4.4a)$$

where D denotes the difference operator

$$D = W(x)(T_{ig} - 1) + W(-x)(T_{-ig} - 1) \quad ((T_{ig}f)(x) := f(x + ig)), \quad (4.4b)$$

with

$$W(x) = \frac{\sin(x + ig_1) \cos(x + ig_2) \sin(x + ig_3) \cos(x + ig_4)}{\sin(2x) \sin(2x + ig)}, \quad (4.4c)$$

and the eigenvalue is given by

$$E_n = (\cosh(\hat{g} + 2ng) - \cosh(\hat{g}))/2, \quad \hat{g} = g_1 + g_2 + g_3 + g_4 - g. \quad (4.4d)$$

After these preliminaries, we are now in the position to prove the main result. It follows from the general fact that the polynomials form an orthogonal system on the interval $(0, \pi/2)$ with respect to a positive weight function that the Askey-Wilson polynomial $p_n(x)$ has n simple zeros inside the interval $(0, \pi/2)$. If we denote these zeros by x_1, \dots, x_n , then it is clear that the Askey-Wilson polynomial factorizes as

$$p_n(x) = 2^{2n-1} \prod_{k=1}^n \sin(x_k + x) \sin(x_k - x) \quad (4.5)$$

(since $2 \sin(x_k + x) \sin(x_k - x) = \cos(2x) - \cos(2x_k)$). After plugging the factorization of the Askey-Wilson polynomial from Eq. (4.5) into the difference equation in Eqs. (4.4a)-(4.4d), and setting of x equal to the j th root x_j , one arrives at the identity

$$\begin{aligned} W(x_j) \prod_{k=1}^n \sin(x_k + x_j + ig) \sin(x_k - x_j - ig) + \\ W(-x_j) \prod_{k=1}^n \sin(x_k + x_j - ig) \sin(x_k - x_j + ig) = 0, \end{aligned} \quad (4.6)$$

which amounts to Eq. (3.1b). This shows that the roots of the Askey-Wilson polynomial solve the nonlinear system of algebraic equations in Eq. (3.1b).

To see that this is the only solution (up to permutation), we now assume—reversely—that the points $0 < x_k < \pi/2$, $k = 1, \dots, n$ are such that they constitute *any* solution to Eq. (3.1b), and show that this implies that the corresponding factorized polynomial of the form $p_n(x)$ (4.5) must be equal to the Askey-Wilson polynomial. To this end it is sufficient to infer that the factorized polynomial in question solves the eigenvalue equation in Eqs. (4.4a)-(4.4d) (since the spectrum of D is nondegenerate as a continuous function of the parameters and thus determines the eigenpolynomials uniquely). It is clear from the fact that the Askey-Wilson polynomials form the corresponding eigenbasis that acting with the operator D

(4.4b), (4.4c) on a monic polynomial of the form in Eq. (4.5) produces the eigenvalue E_n (4.4d) times a certain monic polynomial $q_n(x)$ of degree n . Furthermore, we have that $E_n q_n(x_j) = (Dp_n)(x_j) = 0$ for $j = 1, \dots, n$, because of Eq. (4.6) (which holds since the points x_1, \dots, x_n solve Eq. (3.1b) by assumption). Hence, the monic polynomial $q_n(x)$ has the same roots as $p_n(x)$ and thus coincides with it. In other words, the factorized polynomial $p_n(x)$ solves the Askey-Wilson difference equation, and is thus equal to the Askey-Wilson polynomial, whence the roots x_1, \dots, x_n correspond to the roots of the Askey-Wilson polynomial. This gives rise to the following theorem.

Theorem 4.1. *The unique (up to permutation) solution $0 < x_k < \pi/2$, $k = 1, \dots, n$ of the nonlinear system of algebraic equations in Eq. (3.1b) is given by the (simple) roots of the Askey-Wilson polynomial $p_n(x)$ (4.2) with parameters of the form in Eq. (4.3).*

By combining Theorem 3.1 and Theorem 4.1, we end up with the desired characterization of the ground-state equilibrium configuration in terms of zeros of the Askey-Wilson polynomial.

Corollary 4.2. *The trigonometric BC -type Ruijsenaars-Schneider Hamiltonian $H(\mathbf{p}, \mathbf{x})$ in Eqs. (2.1a)-(2.1c) assumes the global minimum $H = 0$ only at the point in the phase space Ω (2.2) such that $p_1 = p_2 = \dots = p_n = 0$ and $0 < x_1 < x_2 < \dots < x_n < \pi/2$ are given by the (simple) roots of the Askey-Wilson polynomial $p_n(x)$ (4.2) with parameters of the form in Eq. (4.3).*

5. RATIONAL DEGENERATION

By working the way down the Askey hierarchy of (basic) hypergeometric orthogonal polynomials, starting from the Askey-Wilson polynomials corresponding to the trigonometric BC -type Ruijsenaars-Schneider systems, one arrives at the equilibrium configurations associated to the degenerate Ruijsenaars-Schneider systems considered by Sasaki *et al* [11, 9] and at the equilibrium configurations associated to the Calogero-Moser systems considered by Calogero *et al* [2, 3, 4, 10]. As an example, we will wrap up by detailing the important case of the rational BC -type Ruijsenaars-Schneider system [5]. In this case the ground-state equilibrium turns out to be given by the zeros of the Wilson polynomials [9].

The Hamiltonian $H(\mathbf{p}, \mathbf{x})$ of the rational BC -type Ruijsenaars-Schneider system is given by Eqs. (2.1a), (2.1b) with potentials of the form [5]

$$v(x) = (x + ig)/x, \quad w(x) = (x + ig_1)(x + ig_2)(x + ig_3)(x + ig_4)/x^2. \quad (5.1)$$

The phase space becomes in this situation

$$\Omega = \{(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^{2n} \mid 0 < x_1 < x_2 < \dots < x_{n-1} < x_n\}. \quad (5.2)$$

The following theorem characterizes the ground-state equilibrium configuration of the rational BC -type Ruijsenaars-Schneider Hamiltonian in terms of the zeros of the Wilson polynomials [15, 8]

$$p_n(x) = \frac{(-1)^n (a + b, a + c, a + d)_n}{(n + a + b + c + d - 1)_n} \times {}_4F_3 \left[\begin{matrix} -n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} \middle| 1 \right], \quad (5.3a)$$

which satisfy the orthogonality relations

$$\int_0^\infty p_n(x) \overline{p_m(x)} \Delta(x) dx = 0, \quad n \neq m, \quad (5.3b)$$

associated to the positive weight function

$$\Delta(x) = \frac{1}{c(x)c(-x)}, \quad c(x) = \frac{\Gamma(2ix)}{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}, \quad (5.3c)$$

where $a, b, c, d > 0$ (and $\Gamma(\cdot)$ refers to the gamma function).

Theorem 5.1. *The rational BC-type Ruijsenaar-Schneider Hamiltonian $H(\mathbf{p}, \mathbf{x})$ from Eqs. (2.1a), (2.1b), with potentials of the form in Eq. (5.1), has a unique global minimum $H = 0$ in the phase space Ω (5.2) at $p_1 = p_2 = \dots = p_n = 0$ and $0 < x_1 < x_2 < \dots < x_n$ given by the (simple) roots of the rescaled Wilson polynomial $p_n(x/g)$ (5.3a) with rescaled parameters $a = g_1/g$, $b = g_2/g$, $c = g_3/g$, $d = g_4/g$; these roots in turn constitute the unique positive solution to the nonlinear system of algebraic equations*

$$\prod_{1 \leq k \leq n, k \neq j} \frac{(x_j + x_k + ig)(x_j - x_k + ig)}{(x_j + x_k - ig)(x_j - x_k - ig)} = \frac{(x_j - ig_1)(x_j - ig_2)(x_j - ig_3)(x_j - ig_4)}{(x_j + ig_1)(x_j + ig_2)(x_j + ig_3)(x_j + ig_4)},$$

$j = 1, \dots, n.$

It is clear from the theorem that varying the value of the coupling parameter g gives rise to a linear rescaling of the equilibrium positions. More specifically, starting from the $g = 1$ configuration corresponding to the zeros x_1, \dots, x_n of the Wilson polynomials $p_n(x)$ (5.3a) with $a = g_1$, $b = g_2$, $c = g_3$, $d = g_4$, one passes to the equilibrium configuration for general positive g via the rescaling $x_j \rightarrow gx_j$, $j = 1, \dots, n$.

The proof of the above theorem runs along the same lines of Sections 3 and 4, and hinges on the second-order difference equation for the rescaled Wilson polynomials of the form in Eqs. (4.4a), (4.4b) with

$$W(x) = \frac{(x + ig_1)(x + ig_2)(x + ig_3)(x + ig_4)}{2x(2x + ig)}, \quad E_n = -ng(ng + \hat{g}) \quad (5.4)$$

(cf. e.g. [8]). Indeed, it is immediate that the minimization condition $V_j(\mathbf{x}) = V_j(-\mathbf{x})$ now gives rise to the algebraic equations for the equilibrium points stated in the theorem; furthermore, the solution of this system readily follows upon substitution of the factorization $p_n(x) = \prod_{k=1}^n (x + x_k)(x - x_k)$ into the difference equation for the rescaled Wilson polynomials.

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